

Problem Set 3

This third problem set explores graphs, relations, functions, cardinalities, and the pigeonhole principle. This should be a great way to get a feel for how we define mathematical structures and what some of the consequences of those definitions are.

Start this problem set early. It contains seven problems (plus one checkpoint question, one survey question and one extra-credit problem), several of which require a fair amount of thought. I would suggest reading through this problem set at least once as soon as you get it to get a sense of what it covers.

As much as you possibly can, please try to work on this problem set individually. That said, if you do work with others, please be sure to cite who you are working with and on what problems. For more details, see the section on the honor code in the course information handout.

In any question that asks for a proof, you **must** provide a rigorous mathematical proof. You cannot draw a picture or argue by intuition. You should, at the very least, state what type of proof you are using, and (if proceeding by contradiction, contrapositive, or induction) state exactly what it is that you are trying to show. If we specify that a proof must be done a certain way, you must use that particular proof technique; otherwise you may prove the result however you wish.

If you are asked to prove something by induction, you may use either weak induction or strong induction. You should state your base case before you prove it, and should state what the inductive hypothesis is before you prove the inductive step.

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

This problem set has 150 possible points. It is weighted at 7% of your total grade. The earlier questions serve as a warm-up for the later problems, so do be aware that the difficulty of the problems does increase over the course of this problem set.

Good luck, and have fun!

Checkpoint due Monday, April 23 at 2:15PM

Assignment due Friday, April 27 at 2:15PM

Checkpoint Problem: Inverse Relations (25 Points if Submitted)

Write your solutions to the following problems and submit them by this Monday, April 23th at the start of class. These problems will be graded based on whether or not you submit it, rather than the correctness of your solutions. We will try to get these problems returned to you with feedback on your proof style this Wednesday, April 25th. Submission instructions are on the last page of this problem set.

Please make the best effort you can when solving these problems. We want the feedback we give you on your solutions to be as useful as possible, so the more time and effort you put into them, the better we'll be able to comment on your proof style and technique. **Note that this question has four parts.**

Recall from lecture that if R is a binary relation over a set A , the relation R^{-1} is the binary relation

$$R^{-1} = \{ (a, b) \mid bRa \}$$

In other words, $bR^{-1}a$ iff aRb .

- i. What three properties must a relation satisfy to be an equivalence relation?
- ii. Prove or disprove: If R is an equivalence relation, then R^{-1} is an equivalence relation.
- iii. What three properties must a relation satisfy to be a partial order?
- iv. Prove or disprove: If R is a partial order, then R^{-1} is a partial order.

The rest of these problems should be completed and submitted for credit by Friday, April 27.

Problem One: Lexicographical Orderings (20 points)

Suppose that $<_A$ is a strict order over a set A and $<_B$ is a strict order over a set B . How might we combine these strict orders together to get a strict order over pairs in $A \times B$? One option is to use the **lexicographical ordering**, denoted $<_{\text{lex}}$, which is defined as follows. For any $(a_1, b_1) \in A \times B$ and $(a_2, b_2) \in A \times B$:

- If $a_1 <_A a_2$, then $(a_1, b_1) <_{\text{lex}} (a_2, b_2)$.
- Otherwise, if $a_1 = a_2$ and $b_1 <_B b_2$, then $(a_1, b_1) <_{\text{lex}} (a_2, b_2)$

Intuitively, you can think of this as follows. Compare the first elements of the two pairs. If the first element of the first pair is less than the first element of the second pair, then the first pair is less than the second pair. Otherwise, if the first elements are equal, then look at the second elements of each pair. If the second element of the first pair is less than the second element of the second pair, then the first pair is less than the second pair.

Prove that for any strict order $<_A$ over a set A and strict order $<_B$ over a set B , that $<_{\text{lex}}$ is a strict order.

Problem Two: The Cardinality of the Continuum (8 Points)

Cantor's diagonal argument was originally used to prove that $|\mathbb{N}| < |\mathbb{R}|$; that is, that there are strictly fewer natural numbers than real numbers. This came as a complete shock to the mathematical world, since it had always been assumed that these two sets, being infinite, should have the same size.

Recall from lecture that $|\mathbb{Z}| = |\mathbb{N}|$. Accordingly, we know that $|\mathbb{Z}| < |\mathbb{R}|$. Consequently, we should not be able to prove that $|\mathbb{R}| = |\mathbb{Z}|$. But below, we have a proof of just this fact:

Theorem: $|\mathbb{R}| = |\mathbb{Z}|$.

Proof: We exhibit a bijection $f: \mathbb{R} \rightarrow \mathbb{Z}$. Let $f(x) = x$. We prove that f is injective and surjective.

To show that f is injective, consider any x_0, x_1 such that $f(x_0) = f(x_1)$. We will prove that $x_0 = x_1$. To see this, note that by our definition of f , since $f(x_0) = f(x_1)$, we have that $x_0 = x_1$, as required. Thus f is injective.

To show that f is surjective, consider any $x \in \mathbb{Z}$. We need to show that there is an $r \in \mathbb{R}$ such that $f(r) = x$. Since $x \in \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{R}$, we know that $x \in \mathbb{R}$ as well. Thus if we take $r = x$, then $f(r) = r = x$ as required. Thus f is surjective.

Since f is injective and surjective, it is a bijection. Thus, by definition, $|\mathbb{R}| = |\mathbb{Z}|$. ■

Of course, this proof is incorrect and contains a fatal flaw. What's wrong with this proof?

Problem Three: Set Cardinalities (16 Points)

For each of the following, show that the indicated sets have the same cardinality by finding a bijection between them, then proving that your function is a bijection. To prove that your function is a bijection, you can either show that it is injective and surjective, or can show that its inverse is a function.

- i. If $|A| = |C|$ and $|B| = |D|$, prove that $|A \times B| = |C \times D|$. As a hint, since you know that $|A| = |C|$, there is a bijection $f: A \rightarrow C$. There is a similar bijection $g: B \rightarrow D$.
- ii. For any natural number $n \geq 1$, let $S_n = \{k \in \mathbb{N} \mid k < n\}$. For example, $S_1 = \{0\}$, $S_3 = \{0, 1, 2\}$, etc. Prove that $|\mathbb{N} \times S_n| = |\mathbb{N}|$.

Combining your results from (i) and (ii), you have just shown that for any nonempty finite set A , that $|\mathbb{N} \times A| = |\mathbb{N}|$. Intuitively, this means that if you make any (nonzero) finite number of copies of the natural numbers, you end up with exactly the same number of elements as you started. Weird, isn't it?

Problem Four: Why Not Surjections? (8 Points)

In lecture, we defined $|A| \leq |B|$ iff there is an injection $f: A \rightarrow B$. Intuitively, this means that we can pair up the elements of A and B so that we don't double-count any elements of B and don't run out of elements of B .

Another reasonable-sounding definition of $|A| \leq |B|$ would be to say that $|A| \leq |B|$ iff there is a surjection $g: B \rightarrow A$. Intuitively, this means that we can cover all of the elements of A using elements of B without running out of elements of B . However, this definition is not as robust as the definition involving injections. In particular, it is possible to find sets A and B for which A is "obviously" smaller than B , but for which there is no surjection from B onto A .

Find an example of two sets A and B where there is an injection $f: A \rightarrow B$, but for which there is no surjection $g: B \rightarrow A$. You should justify your answer, but you don't need to formally prove it.

Problem Five: Infinite Sequences (24 Points)

An **infinite binary sequence** is an infinite series of 0s and 1s. For example, we can consider the infinite sequence of all zeros (00000...), the infinite sequence of all ones (11111...), or other sequences like these:

```
11011100101110111...
00110101000101000...
11110100100001000...
```

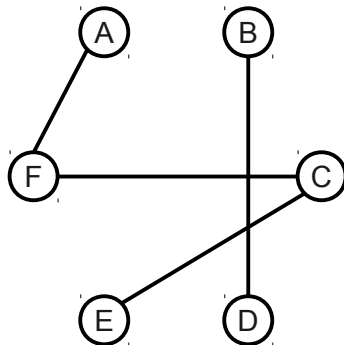
Let \mathbb{B}^ω denote the set of all all infinite binary sequences. In this problem, you will prove $|\mathbb{N}| < |\mathbb{B}^\omega|$, meaning that there are strictly more infinite binary sequences than there are natural numbers.

- i. Prove that $|\mathbb{N}| \leq |\mathbb{B}^\omega|$ by finding an injection $f: \mathbb{N} \rightarrow \mathbb{B}^\omega$.
- ii. Prove that $|\mathbb{N}| \neq |\mathbb{B}^\omega|$ using a proof by diagonalization, similar to the formal proof that we used to prove that $|S| \neq |\wp(S)|$ for any S . While it might help to draw a picture here, you should write a formal mathematical proof here in the style of the one we wrote in lecture.

Problem Six: Pigeonhole Party! (20 Points)

The pigeonhole principle can be used to prove some surprising results about groups of people.

Suppose that you are at a party. Any two people either have met (they are *acquaintances*) or have never met (they are *strangers*). We can therefore think of the party as an undirected graph where each node is a person and each edge connects a pair of acquaintances. For example, at this party:



Person A just knows person F, person B knows person D, and person C knows both person E and person F. However, none of A, B, or E know each other.

- i. Show that at a party with at least two people present, there are at least two people with the same number of acquaintances at that party. (*Hint: Consider two cases: the case where someone knows no one else, and the case where everyone knows at least one person*)

Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x , so $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil \pi \rceil = 4$. The **generalized pigeonhole principle** says that if there are n objects to be put into k boxes, then there must be some box that contains at least $\lceil n / k \rceil$ objects.*

- ii. Using the generalized pigeonhole principle, show that in any group of six people that there are at least three mutual acquaintances or at least three mutual strangers. Three people are mutual acquaintances if each of them knows the other two, and three people are mutual strangers if each person does not know the other two. For example, in the above graph, A, B, and E are mutual strangers, but A, F, and C are **not** mutual acquaintances. (*Hint: Choose any person, then think about how the other people at the party are related to that person*)

Problem Seven: DAGs (24 Points)

Recall from lecture that a directed acyclic graph (DAG) is a directed graph that contains no cycles. A **finite DAG** is a DAG that has finitely many nodes.

In lecture, we discussed the topological sort algorithm, which lists the nodes in a DAG such that no node is listed before all nodes that it has edges into. As part of the algorithm, we assumed that at each step, it was possible to find some node in the DAG that has no outgoing edges. To be mathematically rigorous, we need to prove that this is actually correct.

- i. Prove that in any finite DAG, there must be at least one node with no outgoing edges.
- ii. Prove or disprove: If a graph satisfies property (i), it is a finite DAG.

For future reference, the term “graph” usually refers to finite graphs when no clarification is given.

* I'm not sure why you would be trying to put a whole bunch of pigeons into a small number of pigeonholes, but this result says what would happen if you did. Be nice to animals, folks.

Problem Eight: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

- i. How hard did you find this problem set? How long did it take you to finish?
- ii. Does that seem unreasonably difficult or time-consuming for a five-unit class?
- iii. Did you attend Monday's problem session? If so, did you find it useful?
- iv. How is the pace of this course so far? Too slow? Too fast? Just right?
- v. Is there anything in particular we could do better? Is there anything in particular that you think we're doing well?

Submission instructions

There are three ways to submit this assignment:

1. Hand in a physical copy of your answers at the start of class. This is probably the easiest way to submit if you are on campus.
2. Submit a physical copy of your answers in the filing cabinet in the open space near the handout hangout in the Gates building. If you haven't been there before, it's right inside the entrance labeled "Stanford Engineering Venture Fund Laboratories." There will be a clearly-labeled filing cabinet where you can submit your solutions.
3. Send an email with an electronic copy of your answers to the submission mailing list (cs103-spr1112-submissions@lists.stanford.edu) with the string "[PS3]" somewhere in the subject line.

If you are an SCPD student, we would strongly prefer that you submit solutions via email, especially for the checkpoint problems, so that we can get your solution graded and returned as quickly as possible. Please contact us if this will be a problem.

Extra Credit Problem: Complex Numbers (5 Points Extra Credit)

A *complex number* is a number expressible as $a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and $i = \sqrt{-1}$. The set of all complex numbers is denoted \mathbb{C} .

Prove that $|\mathbb{R}| = |\mathbb{C}|$.